

ON SOME INTEGRAL INEQUALITIES VIA h -CONVEXITY

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ABSTRACT. In this paper, we establish some new inequalities for class of $SX(h, I)$ convex functions which are supermultiplicative or superadditive and nonnegative. And we also give some applications for special means.

1. INTRODUCTION

The following definition is well known in the literature [7]: A function $f : I \rightarrow \mathbb{R}, \emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if inequality

$$(1.1) \quad f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$. Geometrically, this means that if P, Q and R are three distinct points on the graph of f with Q between P and R , then Q is on or below chord PR .

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. The following double inequality:

$$(1.2) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}$$

is known in the literature as Hadamard's inequality for convex function. Keep in mind that some of the classical inequalities for means can come from (1.2) for convenient particular selections of the function f . If f is concave, this double inequality hold in the inversed way.

The inequalities (1.2) which have numerous uses in a variety of settings, has been came a significant groundwork in mathematical analysis and optimization. Many reports have provided new proof, extensions and considering its refinements, generalizations, numerous interpolations and applications, for example, in the theory of special means and information theory. For some results on generalizations, extensions and applications of the Hermite-Hadamard inequalities, see [4, 5, 6, 9-14].

Definition 1. [5] We say that $f : I \rightarrow \mathbb{R}$ is Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0, 1)$ we have

$$(1.3) \quad f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}.$$

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Definition 2. [4] We say that $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a P -function or that f belongs to the class $P(I)$ if f is nonnegative and for all $x, y \in I$ and $t \in [0, 1]$, we have

$$(1.4) \quad f(tx + (1-t)y) \leq f(x) + f(y).$$

Definition 3. [6] Let $s \in (0, 1]$. A function $f : (0, \infty] \rightarrow [0, \infty]$ is said to be s -convex in the second sense if

$$(1.5) \quad f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y),$$

for all $x, y \in (0, b]$ and $t \in [0, 1]$. This class of s -convex functions is usually denoted by K_s^2 .

In 1978, Breckner introduced s -convex functions as a generalization of convex functions in [2]. Also, in that work Breckner proved the important fact that the set valued map is s -convex only if the associated support function is s -convex function in [3]. A number of properties and connections with s -convex in the first sense are discussed in paper [6]. Of course, s -convexity means just convexity when $s = 1$.

Definition 4. [9] Let $h : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a positive function. We say that $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is h -convex function, or that f belongs to the class $SX(h, I)$, if f is nonnegative and for all $x, y \in I$ and $t \in [0, 1]$ we have

$$(1.6) \quad f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y).$$

If inequality (1.6) is reversed, then f is said to be h -concave, i.e. $f \in SV(h, I)$. Obviously, if $h(t) = t$, then all nonnegative convex functions belong to $SX(h, I)$ and all nonnegative concave functions belong to $SV(h, I)$; if $h(t) = \frac{1}{t}$, then $SX(h, I) = Q(I)$; if $h(t) = 1$, then $SX(h, I) \supseteq P(I)$; and if $h(t) = t^s$, where $s \in (0, 1)$, then $SX(h, I) \supseteq K_s^2$.

Definition 5. [9] A function $h : J \rightarrow \mathbb{R}$ is said to be a supermultiplicative function if

$$(1.7) \quad h(xy) \geq h(x)h(y)$$

for all $x, y \in J$.

If inequality (1.7) is reversed, then h is said to be a submultiplicative function. If equality held in (1.7), then h is said to be a multiplicative function.

Definition 6. [1] A function $h : J \rightarrow \mathbb{R}$ is said to be a superadditive function if

$$(1.8) \quad h(x+y) \geq h(x) + h(y)$$

for all $x, y \in J$.

Definition 7. [8] Two functions $h : X \rightarrow \mathbb{R}$ and $g : X \rightarrow \mathbb{R}$ are said to be similarly ordered, shortly f s.o. g , if

$$(1.9) \quad (f(x) - f(y))(g(x) - g(y)) \geq 0$$

for every $x, y \in X$.

Remark 1. [9] Let h be a non-negative function such that

$$(1.10) \quad h(\alpha) \geq \alpha$$

for all $\alpha \in (0, 1)$. For example, the function $h_k(x) = x^k$ where $k \leq 1$ and $x > 0$ has that property. If f is a non-negative convex function on I , then for $x, y \in I$, $\alpha \in (0, 1)$ we have

$$(1.11) \quad f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y) \leq h(\alpha)f(x) + h(1 - \alpha)f(y).$$

So, $f \in SX(h, I)$. Similarly, if the function h has the property: $h(\alpha) \leq \alpha$ for all $\alpha \in (0, 1)$, then any non-negative concave function f belongs to the class $SV(h, I)$.

Proposition 1. [9] Let f and g be a similarly ordered functions on I , i.e.

$$(1.12) \quad (f(x) - f(y))(g(x) - g(y)) \geq 0,$$

for all $x, y \in I$. If $f \in SX(h_1, I)$, $g \in SX(h_2, I)$ and $h(\alpha) + h(1 - \alpha) \leq c$ for all $\alpha \in (0, 1)$, where $h(t) = \max\{h_1(t), h_2(t)\}$ and c is a fixed positive number, then the product fg belongs to $SX(ch, I)$. If f and g are oppositely ordered, $f \in SV(h_1, I)$, $g \in SV(h_2, I)$ and $h(\alpha) + h(1 - \alpha) \geq c$ for all $\alpha \in (0, 1)$, where $h(t) = \min\{h_1(t), h_2(t)\}$ and $c > 0$, then the product fg belongs to $SV(ch, I)$.

Up until now, there are many reports on the two convex function, two s -convex functions, two m -convex functions or on the product of the s -convex function with an ordinary convex function. And in this study, in addition to its predecessors, the new inequalities on the product of classes of h -convex function will be obtained by using the elementary analysis and the applications in the special means for the obtained inequalities will be provided. In this paper we will imply $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

2. MAIN RESULTS

The following inequalities is well known in the literature; For $\mu \leq \lambda$ and $\kappa \leq \varepsilon$ and $\mu, \lambda, \kappa, \varepsilon \in \mathbb{R}$,

$$(2.1) \quad \mu\varepsilon + \lambda\kappa \leq \mu\kappa + \lambda\varepsilon$$

$$(2.2) \quad \mu\kappa \leq \lambda\varepsilon$$

The inequality (2.1) is more useful than the inequality (2.2). In our proofs, the inequality (2.1) will be used.

Theorem 1. Let $f, g \in SX(h, I)$, h is super-multiplicative and f, g be similarly ordered functions on I for all $x, y \in I \subseteq \mathbb{R}$, and $(f, g)(x) \geq x$ and $h(t) \geq t$. Then for all $t \in [0, 1]$, we have;

$$(2.3) \quad \begin{aligned} & \frac{2a + 3b}{6} (f(a) + g(a)) + \frac{3a + 2b}{6} (f(b) + g(b)) \\ & \leq \frac{1}{b - a} \int_a^b (fg)(x) dx + (fg)(a) \int_0^1 [h(t(1 - t)) + h(t^2)] dt \\ & \quad + (fg)(b) \int_0^1 [h(t(1 - t)) + h((1 - t)^2)] dt. \end{aligned}$$

Proof. Since f, g are h -convex functions on I , we have

$$\begin{aligned} f(\alpha x + \beta y) &\leq h(\alpha) f(x) + h(\beta) f(y) \\ g(\alpha x + \beta y) &\leq h(\alpha) g(x) + h(\beta) g(y) \end{aligned}$$

for all $\alpha, \beta \in (0, 1)$, $\alpha + \beta = 1$. Using the elementary inequality $\mu \leq \lambda$ and $\kappa \leq \varepsilon$ then $\mu\varepsilon + \lambda\kappa \leq \mu\kappa + \lambda\varepsilon$ for $\mu, \lambda, \kappa, \varepsilon \in \mathbb{R}$, so we get

$$\begin{aligned} &f(\alpha x + \beta y) [h(\alpha) g(x) + h(\beta) g(y)] \\ &+ g(\alpha x + \beta y) [h(\alpha) f(x) + h(\beta) f(y)] \\ \leq &f(\alpha x + \beta y) g(\alpha x + \beta y) \\ (2.4) \quad &+ [h(\alpha) f(x) + h(\beta) f(y)] [h(\alpha) g(x) + h(\beta) g(y)] \end{aligned}$$

Using the other properties of f, g and h in Theorem 1, we get

$$\begin{aligned} &f(\alpha x + \beta y) [h(\alpha) g(x) + h(\beta) g(y)] \\ &+ g(\alpha x + \beta y) [h(\alpha) f(x) + h(\beta) f(y)] \\ = &h(\alpha) g(x) f(\alpha x + \beta y) + h(\beta) g(y) f(\alpha x + \beta y) \\ &+ h(\alpha) f(x) g(\alpha x + \beta y) + h(\beta) f(y) g(\alpha x + \beta y), \end{aligned}$$

since f, g and h are nonnegative functions

$$\begin{aligned} &h(\alpha) [g(x) f(\alpha x + \beta y) + f(x) g(\alpha x + \beta y)] \\ &+ h(\beta) [g(y) f(\alpha x + \beta y) + f(y) g(\alpha x + \beta y)] \\ \geq &\alpha [(\alpha x + \beta y) g(x) + (\alpha x + \beta y) f(x)] \\ &+ \beta [(\alpha x + \beta y) g(y) + (\alpha x + \beta y) f(y)] \\ (2.5) \quad = &\alpha [(\alpha x + \beta y) (g(x) + f(x))] + \beta [(\alpha x + \beta y) (g(y) + f(y))]. \end{aligned}$$

For the right hand side of (2.4), again using the other properties of f, g and h in Theorem 1, we can write

$$\begin{aligned} &f(\alpha x + \beta y) g(\alpha x + \beta y) \\ &+ [h(\alpha) f(x) + h(\beta) f(y)] [h(\alpha) g(x) + h(\beta) g(y)] \\ = &f(\alpha x + \beta y) g(\alpha x + \beta y) + h^2(\alpha) f(x) g(x) + h^2(\beta) f(y) g(y) \\ &+ h(\alpha) h(\beta) f(x) g(y) + h(\alpha) h(\beta) f(y) g(x) \\ \leq &f(\alpha x + \beta y) g(\alpha x + \beta y) + h^2(\alpha) f(x) g(x) + h^2(\beta) f(y) g(y) \\ &+ h(\alpha\beta) f(x) g(y) + h(\alpha\beta) f(y) g(x) \\ = &f(\alpha x + \beta y) g(\alpha x + \beta y) + h^2(\alpha) f(x) g(x) + h^2(\beta) f(y) g(y) \\ &+ h(\alpha\beta) (f(x) g(y) + f(y) g(x)) \\ \leq &f(\alpha x + \beta y) g(\alpha x + \beta y) + h^2(\alpha) f(x) g(x) + h^2(\beta) f(y) g(y) \\ &+ h(\alpha\beta) (f(x) g(x) + f(y) g(y)) \\ = &f(\alpha x + \beta y) g(\alpha x + \beta y) + [h^2(\alpha) + h(\alpha\beta)] f(x) g(x) \\ (2.6) \quad &+ [h^2(\beta) + h(\alpha\beta)] f(y) g(y) \end{aligned}$$

Now by combining expression (2.5) and (2.6), we obtain

$$\begin{aligned} &\alpha [(\alpha x + \beta y) (g(x) + f(x))] + \beta [(\alpha x + \beta y) (g(y) + f(y))] \\ (2.7) \quad \leq &f(\alpha x + \beta y) g(\alpha x + \beta y) + [h^2(\alpha) + h(\alpha\beta)] f(x) g(x) \\ &+ [h^2(\beta) + h(\alpha\beta)] f(y) g(y). \end{aligned}$$

If we choose $x = a$, $y = b$ and $\beta = 1 - \alpha$ in (2.7), we have

$$\begin{aligned}
 & (\alpha^2 a + (1 - \alpha) b) (f(a) + g(a)) + (\alpha a + (1 - \alpha)^2 b) (f(b) + g(b)) \\
 \leq & f(\alpha a + (1 - \alpha) b) g(\alpha a + (1 - \alpha) b) + [h^2(\alpha) + h(\alpha(1 - \alpha))] f(a) g(a) \\
 & + [h^2(1 - \alpha) + h(\alpha(1 - \alpha))] f(b) g(b) \\
 & \text{by multiplicativity of } h, \text{ we deduce} \\
 \leq & f(\alpha a + (1 - \alpha) b) g(\alpha a + (1 - \alpha) b) + [h(\alpha^2) + h(\alpha - \alpha^2)] f(a) g(a) \\
 (2.8) \quad & + [h((1 - \alpha)^2) + h(\alpha - \alpha^2)] f(b) g(b)
 \end{aligned}$$

By integrating the result with respect to α over $[0, 1]$, we obtain

$$\begin{aligned}
 & \left(a \int_0^1 \alpha^2 d\alpha + b \int_0^1 (1 - \alpha) d\alpha \right) (f(a) + g(a)) \\
 & + \left(a \int_0^1 \alpha d\alpha + b \int_0^1 (1 - \alpha)^2 d\alpha \right) (f(b) + g(b)) \\
 = & \frac{2a + 3b}{6} (f(a) + g(a)) + \frac{3a + 2b}{6} (f(b) + g(b)) \\
 \leq & \frac{1}{b - a} \int_a^b f g(x) dx + f(a) g(a) \int_0^1 [h(\alpha^2) + h(\alpha - \alpha^2)] d\alpha \\
 & + f(b) g(b) \int_0^1 [h((1 - \alpha)^2) + h(\alpha - \alpha^2)] d\alpha
 \end{aligned}$$

which completes the proof. \square

Corollary 1. *If in (2.3) we take $h(t) = 1$, then we get an integral inequality for P -functions with launching of necessary mathematical operations,*

$$\begin{aligned}
 & \frac{2a + 3b}{6} (f(a) + g(a)) + \frac{3a + 2b}{6} (f(b) + g(b)) \\
 \leq & \frac{1}{b - a} \int_a^b (fg)(x) dx + 2M(a, b).
 \end{aligned}$$

Corollary 2. *If in (2.3) we take $h(t) = t$, then we get an integral inequality for ordinary convex functions with launching of necessary mathematical operations,*

$$\begin{aligned}
 & \frac{2a + 3b}{6} (f(a) + g(a)) + \frac{3a + 2b}{6} (f(b) + g(b)) \\
 \leq & \frac{1}{b - a} \int_a^b (fg)(x) dx + (fg)(a) \int_0^1 [(t - t^2) + (t^2)] dt \\
 & + (fg)(b) \int_0^1 [(t - t^2) + ((1 - t)^2)] dt \\
 = & \frac{1}{b - a} \int_a^b (fg)(x) dx + \frac{M(a, b)}{2}.
 \end{aligned}$$

Corollary 3. *If in (2.3) we take $h(t) = t^s$, then we obtain an integral inequality for s -convex functions in the second sense with use of the Beta function of Euler type*

$$\begin{aligned}
& \frac{2a+3b}{6} (f(a) + g(a)) + \frac{3a+2b}{6} (f(b) + g(b)) \\
\leq & \frac{1}{b-a} \int_a^b (fg)(x) dx + (fg)(a) \int_0^1 [t^s(1-t)^s + t^{2s}] dt \\
& + (fg)(b) \int_0^1 [t^s(1-t)^s + (1-t)^{2s}] dt \\
= & \frac{1}{b-a} \int_a^b (fg)(x) dx + (fg)(a) [\beta(s+1, s+1) + \beta(2s+1, 1)] \\
& + (fg)(b) [\beta(s+1, s+1) + \beta(1, 2s+1)] \\
= & \frac{1}{b-a} \int_a^b (fg)(x) dx + M(a, b) [\beta(s+1, s+1) + \beta(2s+1, 1)].
\end{aligned}$$

Theorem 2. *Let $f, g \in SX(h, I)$, h is superadditive and nonnegative such that $h(\alpha) \geq \alpha$ and f, g be similarly ordered functions on I for all $x, y \in I \subseteq \mathbb{R}$. Then for all $\alpha \in (0, 1)$ and $\alpha + \beta = 1$ we have following inequality;*

$$\begin{aligned}
& \frac{M(a, b)}{6} + \frac{N(a, b)}{3} \\
(2.9) \quad & \leq h(1) \left[f(a)g(a) \int_0^1 h(\alpha) d\alpha + f(b)g(b) \int_0^1 h(1-\alpha) d\alpha \right].
\end{aligned}$$

Proof. Since f, g are h -convex functions on I , and using right hand side of (1.11), we have

$$\begin{aligned}
\alpha f(x) + \beta f(y) & \leq h(\alpha) f(x) + h(\beta) f(y) \\
\alpha g(x) + \beta g(y) & \leq h(\alpha) g(x) + h(\beta) g(y)
\end{aligned}$$

for all $\alpha, \beta \in (0, 1)$, $\alpha + \beta = 1$. Using the elementary inequality $\mu \leq \lambda$ and $\kappa \leq \varepsilon$ then $\mu\varepsilon + \lambda\kappa \leq \mu\kappa + \lambda\varepsilon$ for $\mu, \lambda, \kappa, \varepsilon \in \mathbb{R}$, so we get

$$\begin{aligned}
& [\alpha f(x) + \beta f(y)] [h(\alpha) g(x) + h(\beta) g(y)] \\
& + [\alpha g(x) + \beta g(y)] [h(\alpha) f(x) + h(\beta) f(y)] \\
(2.10) \quad & \leq [\alpha f(x) + \beta f(y)] [\alpha g(x) + \beta g(y)] \\
& + [h(\alpha) f(x) + h(\beta) f(y)] [h(\alpha) g(x) + h(\beta) g(y)].
\end{aligned}$$

Using the other properties of h in Theorem 2 on the left hand side of (2.10), we get

$$\begin{aligned}
& [\alpha f(x) + \beta f(y)] [h(\alpha)g(x) + h(\beta)g(y)] \\
& + [\alpha g(x) + \beta g(y)] [h(\alpha)f(x) + h(\beta)f(y)] \\
= & \alpha h(\alpha)f(x)g(x) + \alpha h(\beta)f(x)g(y) \\
& + \beta h(\alpha)f(y)g(x) + \beta h(\beta)f(y)g(y) \\
& + \alpha h(\alpha)f(x)g(x) + \alpha h(\beta)f(y)g(x) \\
& + \beta h(\alpha)f(x)g(y) + \beta h(\beta)f(y)g(y) \\
= & 2\alpha h(\alpha)f(x)g(x) + \alpha h(\beta)[f(x)g(y) + f(y)g(x)] \\
& + \beta h(\alpha)[f(x)g(y) + f(y)g(x)] \\
& + 2\beta h(\beta)f(y)g(y),
\end{aligned}$$

if h be a non-negative function that $h(\alpha) \geq \alpha$

$$\begin{aligned}
& 2\alpha h(\alpha)f(x)g(x) + 2\beta h(\beta)f(y)g(y) \\
& + [\alpha h(\beta) + \beta h(\alpha)][f(x)g(y) + f(y)g(x)] \\
(2.11) \quad \geq & 2\alpha^2 f(x)g(x) + 2\alpha\beta[f(x)g(y) + f(y)g(x)] + 2\beta^2 f(y)g(y).
\end{aligned}$$

However, using the other properties of f , g and h in Theorem 2 on the right hand side of (2.10), we get

$$\begin{aligned}
& [\alpha f(x) + \beta f(y)] [\alpha g(x) + \beta g(y)] \\
& + [h(\alpha)f(x) + h(\beta)f(y)] [h(\alpha)g(x) + h(\beta)g(y)] \\
= & \alpha^2 f(x)g(x) + \alpha\beta f(x)g(y) \\
& + \alpha\beta f(y)g(x) + \beta^2 f(y)g(y) \\
& + h^2(\alpha)f(x)g(x) + h(\alpha)h(\beta)f(x)g(y) \\
& + h(\alpha)h(\beta)f(y)g(x) + h^2(\beta)f(y)g(y) \\
= & \alpha^2 f(x)g(x) + \beta^2 f(y)g(y) \\
& + \alpha\beta[f(x)g(y) + f(y)g(x)] \\
& + h^2(\alpha)f(x)g(x) + h^2(\beta)f(y)g(y) \\
& + h(\alpha)h(\beta)[f(x)g(y) + f(y)g(x)] \\
\leq & \alpha^2 f(x)g(x) + \beta^2 f(y)g(y) + \alpha\beta[f(x)g(x) + f(y)g(y)] \\
& + h^2(\alpha)f(x)g(x) + h^2(\beta)f(y)g(y) \\
& + h(\alpha)h(\beta)[f(x)g(x) + f(y)g(y)] \\
= & [\alpha^2 + \alpha\beta + h^2(\alpha) + h(\alpha)h(\beta)] f(x)g(x) \\
& + [\alpha\beta + \beta^2 + h(\alpha)h(\beta) + h^2(\beta)] f(y)g(y) \\
\leq & [\alpha^2 + \alpha\beta + h(\alpha)h(\alpha + \beta)] f(x)g(x) \\
& + [\alpha\beta + \beta^2 + h(\beta)h(\alpha + \beta)] f(y)g(y) \\
(2.12) \quad = & [\alpha^2 + \alpha\beta + h(\alpha)h(1)] f(x)g(x) \\
& + [\alpha\beta + \beta^2 + h(\beta)h(1)] f(y)g(y).
\end{aligned}$$

Then by expression (2.11) and (2.12);

$$\begin{aligned}
& \alpha^2 f(x)g(x) + 2\alpha\beta[f(x)g(y) + f(y)g(x)] + \beta^2 f(y)g(y) \\
(2.13) \quad \leq & [\alpha\beta + h(\alpha)h(1)] f(x)g(x) + [\alpha\beta + h(\beta)h(1)] f(y)g(y)
\end{aligned}$$

by taking $x = a$, $y = b$ and $\beta = 1 - \alpha$ in (2.13), we have

$$\begin{aligned} & \alpha^2 f(a) g(a) + 2\alpha(1 - \alpha) [f(a) g(b) + f(b) g(a)] + (1 - \alpha)^2 f(b) g(b) \\ & \leq [\alpha(1 - \alpha) + h(\alpha) h(1)] f(a) g(a) + [\alpha(1 - \alpha) + h(1 - \alpha) h(1)] f(b) g(b) \end{aligned}$$

By integrating the result with respect to α over $[0, 1]$, we get

$$\begin{aligned} & f(a) g(a) \int_0^1 \alpha^2 d\alpha + 2[f(a) g(b) + f(b) g(a)] \int_0^1 \alpha(1 - \alpha) d\alpha \\ & + f(b) g(b) \int_0^1 (1 - \alpha)^2 d\alpha \\ & = \frac{f(a) g(a) + f(b) g(b)}{3} + \frac{f(a) g(b) + f(b) g(a)}{3} \\ & \leq f(a) g(a) \left[\int_0^1 \alpha(1 - \alpha) d\alpha + h(1) \int_0^1 h(\alpha) d\alpha \right] \\ & + f(b) g(b) \left[\int_0^1 \alpha(1 - \alpha) d\alpha + h(1) \int_0^1 h(1 - \alpha) dt \right] \\ & = \frac{f(a) g(a) + f(b) g(b)}{6} \\ & + h(1) \left[f(a) g(a) \int_0^1 h(\alpha) d\alpha + f(b) g(b) \int_0^1 h(1 - \alpha) d\alpha \right] \end{aligned}$$

which completes the proof of (2.9). \square

Corollary 4. *In Theorem 2, if we choose $h(t) = t$, then inequality of (2.9) brings inequality (1.12) down.*

Corollary 5. *In Theorem 2, if we choose $h(t) = t^s$, then we get an integral inequality for s -convex functions in the second sense;*

$$(2.14) \quad \frac{M(a, b)}{6} + \frac{N(a, b)}{3} \leq \frac{M(a, b)}{s+1}.$$

And, in (2.14), if we choose $s = 1$, then the inequality of (2.14) above brings inequality (1.12) down.

Theorem 3. *Let $f, g \in SX(h, I)$, h is superadditive and nonnegative such that $h(\alpha) \geq \alpha$ and f, g be similarly ordered functions on I for all $x, y \in I \subseteq \mathbb{R}$. Then for all $\alpha \in (0, 1)$ and $\alpha + \beta = 1$, we have*

$$\begin{aligned} (2.15) \quad & \frac{M(a, b)}{3} + \frac{N(a, b)}{6} \\ & \leq h(1) \left[f(a) g(a) \int_0^1 h(\alpha) d\alpha + f(b) g(b) \int_0^1 h(1 - \alpha) d\alpha \right]. \end{aligned}$$

Proof. As in the proof of the inequality (2.9), since f, g are h -convex functions on I , and using right hand side of (1.12), we have

$$\begin{aligned} \alpha f(x) + \beta f(y) & \leq h(\alpha) f(x) + h(\beta) f(y) \\ \alpha g(x) + \beta g(y) & \leq h(\alpha) g(x) + h(\beta) g(y) \end{aligned}$$

for all $\alpha, \beta \in (0, 1)$, $\alpha + \beta = 1$. Using the elementary inequality $\mu \leq \lambda$ and $\kappa \leq \varepsilon$ then $\mu\varepsilon + \lambda\kappa \leq \mu\kappa + \lambda\varepsilon$ for $\mu, \lambda, \kappa, \varepsilon \in \mathbb{R}$, so we obtain

$$\begin{aligned}
 & [\alpha f(x) + \beta f(y)] [h(\alpha)g(x) + h(\beta)g(y)] \\
 & + [\alpha g(x) + \beta g(y)] [h(\alpha)f(x) + h(\beta)f(y)] \\
 (2.16) \quad & \leq [\alpha f(x) + \beta f(y)] [\alpha g(x) + \beta g(y)] \\
 & + [h(\alpha)f(x) + h(\beta)f(y)] [h(\alpha)g(x) + h(\beta)g(y)].
 \end{aligned}$$

Using the other properties of h in Theorem 3 on the left hand side of (2.16), we get

$$\begin{aligned}
 & [\alpha f(x) + \beta f(y)] [h(\alpha)g(x) + h(\beta)g(y)] \\
 & + [\alpha g(x) + \beta g(y)] [h(\alpha)f(x) + h(\beta)f(y)] \\
 (2.17) \quad & \geq 2\alpha^2 f(x)g(x) + 2\alpha\beta [f(x)g(y) + f(y)g(x)] + 2\beta^2 f(y)g(y)
 \end{aligned}$$

However, using the other properties of f, g and h in Theorem 3 on the right hand side of (2.16), we obtain

$$\begin{aligned}
 (2.18) \quad & [\alpha f(x) + \beta f(y)] [\alpha g(x) + \beta g(y)] \\
 & + [h(\alpha)f(x) + h(\beta)f(y)] [h(\alpha)g(x) + h(\beta)g(y)] \\
 = & \alpha^2 f(x)g(x) + \beta^2 f(y)g(y) + \alpha\beta [f(x)g(y) + f(y)g(x)] \\
 & + h^2(\alpha)f(x)g(x) + h^2(\beta)f(y)g(y) + h(\alpha)h(\beta)[f(x)g(y) + f(y)g(x)] \\
 \leq & \alpha^2 f(x)g(x) + \beta^2 f(y)g(y) + \alpha\beta [f(x)g(x) + f(y)g(y)] \\
 & + h^2(\alpha)f(x)g(x) + h^2(\beta)f(y)g(y) + h(\alpha)h(\beta)[f(x)g(x) + f(y)g(y)] \\
 \leq & h^2(\alpha)f(x)g(x) + h^2(\beta)f(y)g(y) + h(\alpha)h(\beta)[f(x)g(x) + f(y)g(y)] \\
 & + h^2(\alpha)f(x)g(x) + h^2(\beta)f(y)g(y) + h(\alpha)h(\beta)[f(x)g(x) + f(y)g(y)] \\
 = & 2(h^2(\alpha) + h(\alpha)h(\beta))f(x)g(x) + 2(h^2(\beta) + h(\alpha)h(\beta))f(y)g(y) \\
 \leq & 2h(\alpha)h(\alpha + \beta)f(x)g(x) + 2h(\beta)h(\alpha + \beta)f(y)g(y) \\
 = & 2h(\alpha + \beta)[h(\alpha)f(x)g(x) + h(\beta)f(y)g(y)] \\
 = & 2h(1)[h(\alpha)f(x)g(x) + h(\beta)f(y)g(y)]
 \end{aligned}$$

Then by expression (2.17) and (2.18);

$$\begin{aligned}
 & \alpha^2 f(x)g(x) + \alpha\beta [f(x)g(y) + f(y)g(x)] + \beta^2 f(y)g(y) \\
 (2.19) \quad & \leq h(1)[h(\alpha)f(x)g(x) + h(\beta)f(y)g(y)]
 \end{aligned}$$

by taking $x = a, y = b$ and $\beta = 1 - \alpha$ in (2.19), we have

$$\begin{aligned}
 & t^2 f(a)g(a) + t(1-t)[f(a)g(b) + f(b)g(a)] + (1-t)^2 f(b)g(b) \\
 \leq & h(1)[h(\alpha)f(a)g(a) + h(1-\alpha)f(b)g(b)].
 \end{aligned}$$

By integrating the result with respect to α over $[0, 1]$, we obtain

$$\begin{aligned}
& f(a)g(a) \int_0^1 \alpha^2 d\alpha + 2[f(a)g(b) + f(b)g(a)] \int_0^1 \alpha(1-\alpha) d\alpha \\
& + f(b)g(b) \int_0^1 (1-\alpha)^2 d\alpha \\
& = \frac{f(a)g(a) + f(b)g(b)}{3} + \frac{f(a)g(b) + f(b)g(a)}{6} \\
& \leq h(1)[h(\alpha)f(a)g(a) + h(1-\alpha)f(b)g(b)] \\
& = h(1) \left[f(a)g(a) \int_0^1 h(\alpha) d\alpha + f(b)g(b) \int_0^1 h(1-\alpha) d\alpha \right]
\end{aligned}$$

which completes the proof of (2.15). \square

Theorem 4. Let $f, g \in SX(h, I)$, h is superadditive. Then for all $\alpha \in (0, 1)$, we have following inequality;

$$\begin{aligned}
& \left[f\left(\frac{a+b}{2}\right) \frac{g(a)+g(b)}{2} + g\left(\frac{a+b}{2}\right) \frac{f(a)+f(b)}{2} \right] \\
& \times \left[\int_0^1 h(\alpha) d\alpha + \int_0^1 h(1-\alpha) d\alpha \right] \\
(2.20) \quad & \leq (fg)\left(\frac{a+b}{2}\right) + h^2(1)(M(a, b) + N(a, b)).
\end{aligned}$$

Proof. Since $f, g \in SX(h, I)$, we have

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &= f\left(\frac{\alpha a + (1-\alpha)b}{2} + \frac{(1-\alpha)a + \alpha b}{2}\right) \\
&\leq \frac{f(\alpha a + (1-\alpha)b) + f((1-\alpha)a + \alpha b)}{2} \\
&\leq \frac{h(\alpha)f(a) + h(1-\alpha)f(b) + h(1-\alpha)f(a) + h(\alpha)f(b)}{2} \\
&= \frac{[h(\alpha) + h(1-\alpha)][f(a) + f(b)]}{2} \\
g\left(\frac{a+b}{2}\right) &\leq \frac{[h(\alpha) + h(1-\alpha)][g(a) + g(b)]}{2}
\end{aligned}$$

for all $\alpha \in (0, 1)$. Using the elementary inequality $\mu \leq \lambda$ and $\kappa \leq \varepsilon$ then $\mu\varepsilon + \lambda\kappa \leq \mu\kappa + \lambda\varepsilon$ for $\mu, \lambda, \kappa, \varepsilon \in \mathbb{R}$, so we get

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) \frac{[h(\alpha) + h(1-\alpha)][g(a) + g(b)]}{2} \\
& + g\left(\frac{a+b}{2}\right) \frac{[h(\alpha) + h(1-\alpha)][f(a) + f(b)]}{2} \\
& = \left[f\left(\frac{a+b}{2}\right) \frac{g(a) + g(b)}{2} + g\left(\frac{a+b}{2}\right) \frac{f(a) + f(b)}{2} \right] [h(\alpha) + h(1-\alpha)] \\
& \leq f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) + \frac{[h(\alpha) + h(1-\alpha)]^2 [f(a) + f(b)][g(a) + g(b)]}{4} \\
& \leq f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) + \frac{[h(1)]^2 [f(a) + f(b)][g(a) + g(b)]}{4},
\end{aligned}$$

by integrating the result with respect to α over $[0, 1]$, we obtain (2.20). \square

Theorem 5. Let $f, g \in SX(h, I)$, h is supermultiplicative and . Then for all $\alpha \in (0, 1)$, we have following inequality;

$$\begin{aligned}
& \frac{2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)}{M(a, b)} \\
(2.21) \quad & \leq \left[\int_0^1 h(\alpha^2) d\alpha + 2 \int_0^1 h(\alpha - \alpha^2) d\alpha + \int_0^1 h((1-\alpha)^2) d\alpha \right].
\end{aligned}$$

Proof. Since $f, g \in SX(h, I)$, we write

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) & \leq [h(\alpha) + h(1-\alpha)] \frac{f(a) + f(b)}{2} \\
g\left(\frac{a+b}{2}\right) & \leq [h(\alpha) + h(1-\alpha)] \frac{g(a) + g(b)}{2}
\end{aligned}$$

for all $\alpha \in (0, 1)$. Since h is supermultiplicative function and f and g are similarly ordered functions, we get

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
& \leq \frac{[h(\alpha) + h(1-\alpha)]^2}{4} (f(a) + f(b))(g(a) + g(b)) \\
& = \frac{[h^2(\alpha) + 2h(\alpha)h(1-\alpha) + h^2(1-\alpha)]^2}{4} \\
& \quad \times (f(a)g(a) + f(b)g(b) + f(a)g(b) + f(b)g(a)) \\
& \leq \frac{[h(\alpha^2) + 2h(\alpha(1-\alpha)) + h((1-\alpha)^2)]^2}{2} (f(a)g(a) + f(b)g(b))
\end{aligned}$$

by integrating the result with respect to α over $[0, 1]$, we obtain (2.21). \square

Corollary 6. *If in (2.21) we take $h(\alpha) = 1$, then we obtain an integral inequality for P -functions with launching of necessary mathematical operations,*

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq 2M(a, b)$$

Corollary 7. *If in (2.21) we take $h(\alpha) = \alpha$, then we obtain an integral inequality for ordinary convex functions with launching of necessary mathematical operations,*

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{M(a, b)}{2}.$$

Corollary 8. *If in (2.21) we take $h(\alpha) = \alpha^s$, then we obtain an integral inequality for s -convex functions in the second sense with the use of the Beta function of Euler type*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ & \leq \frac{M(a, b)}{2} \left[\int_0^1 \alpha^{2s} d\alpha + 2 \int_0^1 \alpha^s (1-\alpha)^s d\alpha + \int_0^1 (1-\alpha)^{2s} d\alpha \right] \\ & = \frac{M(a, b)}{2} \{ \beta(2s+1, 1) + 2\beta(2s+1, 2s+1) + \beta(1, 2s+1) \} \\ & = M(a, b) \{ \beta(2s+1, 1) + \beta(2s+1, 2s+1) \}. \end{aligned}$$

Theorem 6. *Let $I = [a, b] \subseteq \mathbb{R}$, $f, g \in SX(h, I)$, h is supermultiplicative and f, g are symmetric about $\frac{a+b}{2}$, then for all $x \in I$ and $\alpha \in [0, 1]$, we have the following inequalities;*

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(x)g(x) dx \\ (2.22) \quad & \leq \frac{M(a, b) + N(a, b)}{4} \\ & \times \left[\int_0^1 h(\alpha^2) d\alpha + 2 \int_0^1 h(\alpha(1-\alpha)) d\alpha + \int_0^1 h((1-\alpha)^2) d\alpha \right] \end{aligned}$$

Proof. Since $f, g \in SX(h, I)$, we can write

$$\begin{aligned} f(\alpha a + (1-\alpha)b) + f((1-\alpha)a + \alpha b) & \leq [h(\alpha) + h(1-\alpha)][f(a) + f(b)] \\ g(\alpha a + (1-\alpha)b) + g((1-\alpha)a + \alpha b) & \leq [h(\alpha) + h(1-\alpha)][g(a) + g(b)] \end{aligned}$$

for all $\alpha \in [0, 1]$. Since h is a supermultiplicative function, and f and g are symmetric about $\frac{a+b}{2}$, we get

$$\begin{aligned} & [f(\alpha a + (1-\alpha)b) + f((1-\alpha)a + \alpha b)][g(\alpha a + (1-\alpha)b) + g((1-\alpha)a + \alpha b)] \\ & = 4f(\alpha a + (1-\alpha)b)g(\alpha a + (1-\alpha)b) \\ & \leq [h(\alpha) + h(1-\alpha)]^2 [f(a) + f(b)][g(a) + g(b)] \\ & = [h^2(\alpha) + 2h(\alpha)h(1-\alpha) + h^2(1-\alpha)][M(a, b) + N(a, b)] \\ & \leq [h(\alpha^2) + 2h(\alpha(1-\alpha)) + h((1-\alpha)^2)][M(a, b) + N(a, b)]. \end{aligned}$$

By integrating the result with respect to α over $[0, 1]$, and taking into account the change of variable $x = \alpha a + (1 - \alpha)b$, we obtain

$$\begin{aligned} & \frac{4}{b-a} \int_a^b f(x) g(x) dx \\ & \leq [M(a, b) + N(a, b)] \left[\int_0^1 h(\alpha^2) d\alpha + 2 \int_0^1 h(\alpha(1-\alpha)) d\alpha + \int_0^1 h((1-\alpha)^2) d\alpha \right] \end{aligned}$$

which completes the proof. \square

Corollary 9. *If in (2.22) we take $h(\alpha) = 1$, then we obtain an integral inequality for P -functions with launching of necessary mathematical operations,*

$$\frac{1}{b-a} \int_a^b f(x) g(x) dx \leq M(a, b) + N(a, b).$$

Corollary 10. *If in (2.22) we take $h(\alpha) = \alpha$, then we obtain an integral inequality for ordinary convex functions with the use of the necessary mathematical operations,*

$$\frac{1}{b-a} \int_a^b f(x) g(x) dx \leq \frac{M(a, b) + N(a, b)}{4}.$$

Corollary 11. *If in (2.22) we take $h(\alpha) = \alpha^s$, then we obtain an integral inequality for s -convex functions in the second sense the with use of the Beta function of Euler type*

$$\frac{1}{b-a} \int_a^b f(x) g(x) dx \leq \frac{M(a, b) + N(a, b)}{2} \{\beta(2s+1, 1) + \beta(2s+1, 2s+1)\}.$$

3. APPLICATIONS TO SOME SPECIAL MEANS

We now consider the applications of our Theorems to the following special means

The arithmetic mean: $A = A(a, b) := \frac{a+b}{2}$, $a, b \geq 0$,

The geometric mean: $G = G(a, b) := \sqrt{ab}$, $a, b \geq 0$,

The quadratic mean: $K = K(a, b) := \sqrt{\frac{a^2+b^2}{2}}$, $a, b \geq 0$,

The p -logarithmic mean: $L_p = L_p(a, b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)} \right]^{1/p} & \text{if } a \neq b \\ a & \text{if } a = b \end{cases},$

$p \in \mathbb{R} \setminus \{-1, 0\}$; $a, b > 0$.

The following inequality is well known in the literature:

$$H \leq G \leq L \leq I \leq A \leq K$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_1 = A$, $L_0 = I$ and $L_{-1} = L$.

The following propositions holds:

Proposition 2. *Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{Z}$, $|n| \geq 1$. Then, we have:*

$$(3.1) \quad \begin{aligned} & \frac{4}{3} A(a^{n+1}, b^{n+1}) + 2G^2(a, b) A(a^{n-1}, b^{n-1}) \\ & \leq L_{2n}^{2n}(a, b) + A(a^{2n}, b^{2n}). \end{aligned}$$

Proof. If we apply Theorem 1 for $f(x) = g(x) = x^n$, $h(\alpha) = \alpha$ where $x \in \mathbb{R}$, $n \in \mathbb{Z}$, $|n| \geq 1$, we get the proof (3.1). \square

Proposition 3. *Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{Z}$, $|n| \geq 1$. Then, we have:*

$$(3.2) \quad G^2(a^n, b^n) \leq 2A(a^{2n}, b^{2n}).$$

Proof. The proof is immediate from Theorem 2 applied for $f(x) = g(x) = x^n$, $h(\alpha) = \alpha$ where $x \in \mathbb{R}$, $n \in \mathbb{Z}$, $|n| \geq 1$. \square

Proposition 4. *Let $a, b \in \mathbb{R}$, $0 < a < b$. Then, we have:*

$$(3.3) \quad \frac{1}{G^2(a, b)} \leq K(a, b).$$

Proof. The assertion follows from Theorem 2 applied to $f(x) = g(x) = \frac{1}{x}$, $x \in [a, b]$ and $h(\alpha) = \alpha$. \square

Proposition 5. *Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{Z}$, $|n| \geq 1$. Then, we have:*

$$(3.4) \quad G^2(a^n, b^n) \leq A(a^{2n}, b^{2n})$$

Proof. The assertion follows from Theorem 3 applied to $f(x) = g(x) = \frac{1}{x}$, $x \in [a, b]$ and $h(\alpha) = \alpha$. \square

Proposition 6. *Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{Z}$, $|n| \geq 1$. Then, we have:*

$$(3.5) \quad L_{2n}^{2n}(a, b) \leq \frac{A(a^{2n}, b^{2n}) + G^2(a^n, b^n)}{2}.$$

Proof. If we apply Theorem 6 for $f(x) = g(x) = x^n$, $h(\alpha) = \alpha$ where $x \in \mathbb{R}$, $n \in \mathbb{Z}$, $|n| \geq 1$, we get the proof (3.5). \square

Proposition 7. *Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{Z}$, $|n| \geq 1$. Then, we have:*

$$(3.6) \quad G(a, b) \leq A(a, b)$$

Proof. If we apply Theorem 6 for $f(x) = g(x) = \frac{1}{x}$, $x \in [a, b]$ and $h(\alpha) = \alpha$, we get the proof (3.6). \square

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